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THE WEAK ASYMPTOTIC SOLUTION OF THE PHASE FIELD SYSTEM IN THE CASE OF CONFLUENCE OF FREE BOUNDARIES IN THE STEFAN–GIBBS-THOMSON PROBLEM

Using the weak asymptotics method, we analyze the confluence of free boundaries in the Stefan–Gibbs-Thomson problem. We construct the global in time solution of the phase field system in this case.

1. Introduction.

In this paper, we analyze interaction of free boundaries in the Stefan–Gibbs–Thomson problem that is invariant under the change $x \to -x$. We consider a one-dimensional binary medium that conventionally has "+" and "−" phases. We assume that this medium occupies the domain (interval) $\Omega \in \mathbb{R}^1_x$, $\Omega = [-l, l]$, $l = \text{const}$. Let the two interfaces $\Gamma_{1,t} = \{x;\ x = -\hat{\varphi}(t)\}$ and $\Gamma_{2,t} = \{x;\ x = \hat{\varphi}(t)\}$ exist. So, the interval Ω is divided into the three sections $\Omega^+_{1,t} = [-l, -\hat{\varphi}(t)]$, $\Omega^-_t = (-\hat{\varphi}(t), \hat{\varphi}(t))$, $\Omega^+_{2,t} = [\hat{\varphi}(t), l]$, where $\hat{\varphi}(t)$ is some desired function. We denote $\Omega^+_t = \Omega^+_{1,t} \bigcup \Omega^+_{2,t}$ and, consequently, the domain Ω^+_t corresponds to the phase "+", and the domain Ω^-_t corresponds to the phase "−" at the moment of time t. We base our consideration of this problem on the phase field model. In our case, the phase field system has the form [10]

$$L\theta = -\frac{\partial u}{\partial t}, \tag{1}$$

$$\mathcal{L}u = 0, \tag{2}$$

where

$$L\theta = \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2}, \quad \mathcal{L}u = \varepsilon L u - \frac{u - u^3}{\varepsilon} - \varkappa\theta. \tag{3}$$

Here $x \in \mathbb{R}^1$, $t \in [0, t_1]$ ($t_1 > t^*$, where t^* is the moment of confluence of the free boundaries $\Gamma_{i,t}$, $i = 1, 2$), the function θ is the temperature, u is the order function (the value $u = -1$ corresponds to the phase "−", and the value $u = 1$ corresponds to the phase "+"), $\varkappa$ is some constant, and $\varepsilon \ll 1$ is a small parameter.

We assume that equations (1), (2) are completed with compatible initial and boundary conditions.

It is well known [3], [11] that the weak limit $\overline{u}$, $\overline{\theta}$ of the solution of system (1), (2) (the values $\overline{u} = \pm 1$ correspond to the domains Ω^+_t and Ω^-_t) is described as the solution of the heat equation

$$\frac{\partial \overline{\theta}^{\pm}_i}{\partial t} = \frac{\partial^2 \overline{\theta}^{\pm}_i}{\partial x^2}, \quad x \in \Omega^{\pm}_{i,t}, \quad t \in [0, t^*), \tag{4}$$

supplemented with the conditions on the free boundaries

$$\left[\overline{\theta}^{\pm}\right]\Big|_{x=\pm\hat{\varphi}} = 0, \tag{5}$$

$$\left[\frac{\partial \overline{\theta}^{\pm}}{\partial x}\right]\Big|_{x=\pm\hat{\varphi}} = 2\hat{\varphi}_t, \tag{6}$$

$$\overline{\theta}^{\pm}\Big|_{x=\pm\hat{\varphi}} = -\alpha\hat{\varphi}_t. \tag{7}$$

Here $\overline{\theta}^{+}$ is the temperature in the domain Ω_t^+, $\overline{\theta}^{-}$ is the temperature in the domain Ω_t^- (i.e., $\overline{\theta} = \overline{\theta}^{+}$ for $x \in \Omega_t^+$ and $\overline{\theta} = \overline{\theta}^{-}$ for $x \in \Omega_t^-$), α is some constant, and $[g]|_{\Gamma_{1,2,t}} = g(\mp\hat{\varphi}+0) - g(\mp\hat{\varphi}-0)$ is the jump of the function g on the interfaces $\Gamma_{1,2,t}$.

The asymptotics $\hat{u}_\varepsilon$, $\hat{\theta}_\varepsilon$ of the solution of the phase field system (1), (2) is also expressed in terms of the solution of the Stefan–Gibbs–Thomson problem (4)–(7).

We assume that the initial data for problem (4)–(7) are chosen so that the left boundary $\Gamma_{1,t}$ moves to the right ($-\hat{\varphi}_t > 0$) and the right boundary $\Gamma_{2,t}$ moves to the left ($\hat{\varphi}_t < 0$). We also assume that initial and boundary conditions provide the existence of the function $\hat{\varphi}(t) \in C^1$, and $\overline{\theta}^{\pm}(x,t) \in C^{2,1}$ for $0 < t < t^*$ ($t = t^*$ is the moment of confluence, $\hat{\varphi}(t^*) = 0$). In addition, we assume that the finite limit

$$\lim_{t\to t^*-0} \hat{\varphi}(t) = \text{const} \tag{8}$$

exists.

In the case considered, for $t < t^*$, the asymptotics of system (1), (2) has the form

$$\theta_\varepsilon^{as} = \bar{\theta}^-(x,t) + (\bar{\theta}^+(x,t) - \bar{\theta}^-(x,t))\,\omega_1\left(\frac{-x-\hat{\varphi}(t)}{\varepsilon}\right)\omega_1\left(\frac{x-\hat{\varphi}(t)}{\varepsilon}\right), \tag{9}$$

$$\begin{aligned} u_\varepsilon^{as} =& 1 + \omega_0\left(\frac{-x-\hat{\varphi}(t)}{\varepsilon} - \check{\varphi}\right) + \\ & \omega_0\left(\frac{x-\hat{\varphi}(t)}{\varepsilon} + \check{\varphi}\right) + \varepsilon\left[\frac{\theta_\varepsilon^{as}}{2} + \omega\left(t, \frac{-x-\hat{\varphi}(t)}{\varepsilon}, \frac{x-\hat{\varphi}(t)}{\varepsilon}\right)\right]. \end{aligned} \tag{10}$$

Here $\omega_1(z) \to 0, 1$ as $z \to \pm\infty$, $\omega_1^k(z) \in \mathbb{S}(\mathbb{R}_z^1)$ for $k > 0$, $\check{\varphi}(t)$ is a smooth function. The function $\omega_0(z) = \tanh(z)$ is the solution of the model equation, and $\omega(t, z_1, z_2) \in C^\infty([0,t^*], \mathbb{S}(\mathbb{R}_z^2))$. If an initial data for (1), (2) has form (9), (10) at $t = 0$, then, for $t < t^*$, the estimate

$$\|u - u_\varepsilon^{as}; C(0,T; L^2(\mathbb{R}^1)\| + \|\theta - \theta_\varepsilon^{as}; \mathcal{L}^2(Q)\| \leqslant c\varepsilon^\mu, \quad \mu \geqslant 3/2$$

holds (see [1], [2], [4]). Here $Q = \mathbb{R}^1 \times [0, t^*)$, and the constant c is independent of ε.

The main goal of this paper is to construct the formal asymptotic solution of system (1), (2) describing interaction of free boundaries, i.e. we want to construct some approximating (asymptotic) formula of the solution and this formula must work for $t \in [0, t_1]$, $t_1 > t^*$. According to the general existence theorems, the solution of problem (1), (2) exists in the situation considered. However, any construction of asymptotics for such solution by the traditional methods is very difficult, because, to find the main term of the order function asymptotics in the case of interaction of free boundaries in the frame work of classical asymptotic methods, we need to solve a partial differential equation explicitly. We use another approach that is called the weak asymptotics method [6],[7]–[10].

We note that, as far as we know, the single example of an accurate qualitative analysis of the problem with confluence of free boundaries for the Hele-Shaw problem was done by

A. Meirmanov and B. Zaltzman [12]. Also, using the weak asymptotics method, G. Omel'yanov [7] has analyzed interaction of free boundaries. In comparison with him, we have changed the structure of anzatz for the weak asymptotic solution and this allowed us to consider the problems in more detail.

It should be noted that the price payed for the possibility to construct more or less explicit formulas for the solution is not so small – we do not justify the asymptotics constructed and this justification can not be derived from known estimates.

The main result of our paper is problem (41). The solution of this system allows us to describe the temperature in the domain Ω uniformly in time. Moreover, formulas that are based on system (41) allow us to explain the behaviour of the temperature at the moment of confluence of the free boundaries (see the data of the numerical simulation in Section and Section).

2. Construction of the weak asymptotic solution.

In particular, we need the following definition. A family of functions $f(t,x,\varepsilon)$ admits the estimate $\mathcal{O}_{\mathcal{D}'}(\varepsilon^\nu)$ if, for any test function $\zeta(x)$, the relation

$$\langle f,\zeta\rangle = \mathcal{O}(\varepsilon^\nu)$$

holds, and this estimate is uniform in $t\in[0,t_1]$.

According to the weak asymptotics method, we need to regularize problem (4)–(7). An appropriate regularization is the field phase system (1), (2) (see [1], [2], [4], [11])

DEFINITION 1. *A pair of smooth functions $(\hat u_\varepsilon,\hat\theta_\varepsilon)$ is a weak asymptotic solution of system (1), (2) if, for any test function $\zeta(x)$, the following relations are satisfied:*

$$\int\left(\hat u_{\varepsilon t}+\hat\theta_{\varepsilon t}\right)\zeta dx+\int\hat\theta_{\varepsilon x}\zeta_x dx=\mathcal{O}(\varepsilon) \tag{11}$$

$$\begin{aligned}&\varepsilon\int\hat u_{\varepsilon t}\zeta dx+\frac{\varepsilon}{2}\int\hat u_\varepsilon^2\zeta_x dx-\\&-\frac{1}{\varepsilon}\int\left(\frac{\hat u_\varepsilon^4}{4}-\frac{\hat u_\varepsilon^2}{2}+\frac14\right)\zeta_x dx+\varkappa\int\hat u_\varepsilon\frac{\partial}{\partial x}(\hat\theta_\varepsilon\zeta)dx=\mathcal{O}(\varepsilon).\end{aligned} \tag{12}$$

Here and later all integrals are taken over $\mathbb{R}^1$. Equation (12) is obtained by multiplying (2) by $\hat u_{\varepsilon x}$ and integrating by parts. The reminders $\mathcal{O}(\varepsilon)$ in the right-hand sides of (11) and (12) must locally be bounded in t, i.e. for any $t_1\in[0,\infty)$, we have

$$\max_{0\leqslant t\leqslant t_1}|\mathcal{O}(\varepsilon)|\leqslant C_{t_1}\varepsilon,\quad C_{t_1}=\text{const}.$$

This construction was introduced and analyzed in paper [2].

In the case of the single interface Γ_t, a solution of the Stefan–Gibbs–Thomson problem has the form (see [1], [2],[3])

$$u=\omega_0\left(\beta\frac{S}{\varepsilon}\right)=\text{sign}(S)+\mathcal{O}_{\mathcal{D}'}(\varepsilon^2),$$

$$\theta = \frac{1}{2}\left(\theta_c^+ + \theta_c^-\right) + \frac{1}{2}\left(\theta_c^+ + \theta_c^-\right)\omega_1\left(\beta\frac{S}{\varepsilon}\right) =$$
$$\frac{1}{2}\left(\theta_c^+ + \theta_c^-\right) + \frac{1}{2}\left(\theta_c^+ + \theta_c^-\right)\operatorname{sign}(S) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),$$

where the free boundary Γ_t is defined by the function $S = t + \Phi(x)$ (i.e., $\Gamma_t = \{x : t = -\Phi(x)\}$), $\beta = 1/|\nabla\Phi|$, and $\theta_c^{\pm}$ are smooth continuations of the functions $\theta^{\pm}$ into the domain Ω. Recall, that the functions $\theta^{\pm}$ are the temperature of the medium in the domains with phases "+" and "−" correspondingly.

We note that all our considerations are formal. We do not justify that the functions constructed are close to the exact solution of system (1), (2) for all the moments of time considered. But, as is noted above, the asimptotics was justified for $t < t^*$.

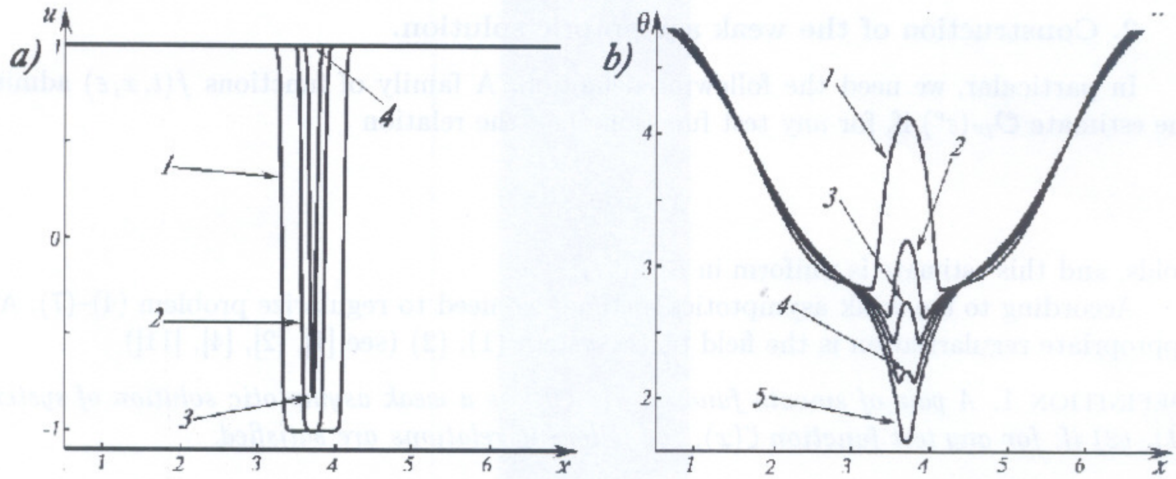

Figure 1. *a)* Confluence of free boundaries. The moments of time are denoted by: 1) t=1.17, 2) t=1.27, 3) t=1.29, 4) $t > t^* \approx 1.31$; *b)* The effect of appearance of a cooled region. The moments of time are denoted by: 1) t=1.17, 2) t=1.21, 3) t=1.27, 4) t=1.30, 5) t=1.31.

The main results of our paper is problem (41) for the weak asymptotic solution of the phase field system specified for $t \geqslant 0$. This system allows us to qualitatively describe the temperature behavior at the moment of confluence of the free boundaries. Namely, from Fig.1 b) we can see that in the Stefan–Gibbs–Thomson problem the temperature has a "jump"with finite amplitude near the moment of confluence: the profile of graphs abruptly changes from W-profile to V-profile.

The evolution of the order function are shown in Fig.1 a). Clearly, in the case of the problem with two free boundaries, prior to the moment of interaction $t = t^*$, the three domains (intervals) exist with different phases. At the moment of contact of the free boundaries the domain (interval) Ω_t^- disappears, and for $t > t^*$ only the single phase "+"exists. It is also clearly seen that problem (1), (2) is essentially nonlinear, and the simple sum of the two waves $\omega_0\left(\frac{-x-\hat{\varphi}(t)}{\varepsilon}\right)$ and $\omega_0\left(\frac{x-\hat{\varphi}(t)}{\varepsilon}\right)$ is not a solution of problem (1), (2). From formula (10) we see that each of these waves has the kink-type structure (see Fig.1 a)). If the free boundaries $\Gamma_{i,t}$, $i = 1, 2$ lie on a sufficient large distance one from another for $t < t^*$, i.e.

$$|\hat{\varphi} - (-\hat{\varphi})| = |2\hat{\varphi}| > \varepsilon^{1-\mu}, \quad \mu > 0,$$

then in (1), (2) the order function has form (10). At the same time, if $t = t^*$ and $\hat{\varphi}_t \neq 0$ then formulas (9), (10) do not give a true solution even in the qualitative sense.

Taking the facts mentioned above into account, we can construct the solution of the phase field system (1), (2) in the form

$$\hat{u}_\varepsilon = \frac{1}{2}\Big[1 + \omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right) + \omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right) - \\ - \omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right)\Big], \tag{13}$$

$$\hat{\theta}_\varepsilon = \gamma(x,t) + T_l^+\omega_1\left(\frac{-x-\varphi}{\varepsilon}\right) + T_r^+\omega_1\left(\frac{x-\varphi}{\varepsilon}\right) + \\ + T^-\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi-x}{\varepsilon}\right), \tag{14}$$

where

$$\gamma(x,t) = \varphi_t(t), \quad T^-(x,t) = \gamma^-(x,t)\frac{(x+\varphi(t,\varepsilon))(\varphi(t,\varepsilon)-x)}{2\varphi(t,\varepsilon)},$$

$$T_l^+(x,t) = \gamma_l^+(x,t)(x+\varphi(t)), \quad T_r^+(x,t) = \gamma_r^+(x,t)(x-\varphi(t)).$$

Here β, φ, γ^-, γ_l^+, γ_r^+ are desired functions, and $\varphi = \varphi(t,\varepsilon)$ is the function described the interfaces $\Gamma_{i,t}$, $i = 1, 2$. To describe a structure of the functions introduced above we consider the "fast"variable τ. Namely, we assume

$$\tau = \frac{\varphi_0(t)}{\varepsilon},$$

where for $t < t^*$ the function $\varphi_0 = \hat{\varphi}$ is found from problem (4)–(7). According to our assumptions, the function $\hat{\varphi}(t)$ can be smoothly continued for $t > t^*$ with preserving the sign of the derivative. This continued function we shall also denote by φ_0. So, the variable $\tau = \tau(t,\varepsilon)$ is specified for $t \geqslant t^*$, and $\tau \to \infty$ for $t < t^*$ as $\varepsilon \to 0$ (prior to interaction of the free boundaries); $\tau \to -\infty$ for $t > t^*$ as $\varepsilon \to 0$ (after confluence of the free boundaries). Hereinafter we assume $\beta = \beta_0 + \beta_1(\tau) > 0$, $\varphi = \varphi_0 + \varphi_0\varphi_1(\tau)$. The functions $\omega_0(z)$, $\omega_1(z)$ are defined in (9), (10). We assume (and we prove later) that the limits $\beta_1(\tau) \to 0$ and $\varphi_1(\tau) \to 0$ hold as $\tau \to \infty$ (i.e., prior to confluence of the free boundaries).

From formula (13) we can see that, as a result of interaction of free boundaries, the kinks $\omega_0(\beta(-x-\varphi)/\varepsilon)$ and $\omega_0(\beta(x-\varphi)/\varepsilon)$ annihilate. This annihilation means the disappearance of the domain Ω_t^-. Indeed, if $\varphi(t) > 0$ then from formula (13) we obtain

$$\hat{u}_\varepsilon = 1 - o(1), \quad x < -\varphi, x > \varphi$$

and

$$\hat{u}_\varepsilon = -1 + o(1), \quad -\varphi < x < \varphi.$$

If $\varphi(t^*) = x^*$ (in our case $x^* = 0$) then

$$\hat{u}_\varepsilon = \left[1 + \omega_0^2(\beta x/\varepsilon)\right]/2,$$

if $\varphi < 0$ then

$$\hat{u}_\varepsilon = 1 - o(\varepsilon^N)$$

for any $N > 0$, $x \in \mathbb{R}^1$. Formula (14) is constructed so that it qualitatively true describes the phases during the evolution of the free boundaries $\Gamma_{i,t}$.

According to Definition 1, to construct a weak asymptotic solution of the phase field system (1), (2) in form (13), (14), we must satisfy relations (11), (12). Here we present only the results of the calculations, and, in Section , we demonstrate the techniques of the weak asymptotics method for (15)–(27) in detail. So, we have

$$\begin{aligned} L\hat{\theta}_\varepsilon + \frac{\partial \hat{u}_\varepsilon}{\partial t} =& W_0\left[H(\varphi - x) - H(-x-\varphi)\right] + \\ & W_1 H(-x-\varphi) + W_2[1 - H(\varphi - x)] + \\ & W_1^1 \delta(x+\varphi) + W_2^1 \delta(\varphi - x) + \\ & W_1^2 \delta'(x+\varphi) + W_2^2 \delta'(\varphi - x) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial \hat{u}_\varepsilon}{\partial x}\mathcal{L}\hat{u}_\varepsilon =& V_1^1\delta(x+\varphi) + V_2^1\delta(\varphi - x) + \\ & V_1^2\delta'(x+\varphi) + V_2^2\delta'(\varphi - x) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \end{aligned} \tag{16}$$

where the operators L and $\mathcal{L}$ are defined by formula (3). In equations (15), (16), the function $H(z)$ is the Heaviside function, and the function $\delta(z)$ is the Dirac δ-function.

In the right-hand side of (15), the coefficients of the Heaviside functions have the form

$$W_0 = B_{\dot{1}1}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(\varphi_t + T^-), \qquad -\varphi < x < \varphi, \tag{17}$$

$$W_1 = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(\varphi_t + T_l^+), \qquad x < -\varphi, \tag{18}$$

$$W_2 = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(\varphi_t + T_r^+), \qquad x > \varphi. \tag{19}$$

In (15), the coefficients of the δ-functions are defined by the formulas

$$W_1^1 = T_{lx}^+\big|_{x=-\varphi} - \gamma^-(-\varphi,t)B_{\dot{1}1} - \frac{\varphi_t}{2}(2 - B_{\dot{0}0}) + \frac{\beta_\tau\varphi_{0t}}{\beta^2}B_{\dot{0}0}^z, \tag{20}$$

$$W_2^1 = -T_{rx}^+\big|_{x=\varphi} - \gamma^-(\varphi,t)B_{\dot{1}1} - \frac{\varphi_t}{2}(2 - B_{\dot{0}0}) + \frac{\beta_\tau\varphi_{0t}}{\beta^2}B_{\dot{0}0}^z. \tag{21}$$

Here we denote

$$B_{\dot{0}0} = \int \dot{\omega}_0(z)\omega_0(-\eta - z)dz, \quad B_{\dot{1}1} = \int \dot{\omega}_1(z)\omega_1(\eta - z)dz,$$

$$B_{\dot{0}0}^z = \int z\dot{\omega}_0(z)\omega_0(-\eta - z)dz, \quad \eta = 2\rho\beta, \qquad \rho = \frac{\varphi}{\varepsilon}.$$

In equation (15), the coefficients of the derivatives of the δ-functions are given by the formulas

$$W_1^2 = T_l^+\big|_{x=-\varphi} = 0, \tag{22}$$

$$W_2^2 = T_r^+\big|_{x=\varphi} = 0. \tag{23}$$

In equation (16), the coefficients of the δ-functions have the form

$$V_1^1 = \frac{\beta_\tau \varphi_{0t}}{2\beta^2}(2B^z_{\dot{0}^2 0} + B^z_{\dot{0}^2 0^2}) + \frac{\beta_\tau \varphi_{0t}}{2\beta^2}(C^z_{\dot{0}\dot{0}} - 2C^z_{\dot{0}\dot{0}0} + C^z_{\dot{0}\dot{0}00}) + \frac{\varkappa\varphi_t}{2}(2 - B_{\dot{0}0}) +$$
$$\frac{\varphi_t}{4}(2a_F - 2B_{\dot{0}^2 0} + B_{\dot{0}^2 0^2}) + \frac{\varphi_t}{8}(C_{\dot{0}\dot{0}} - 2C_{\dot{0}\dot{0}0} + C_{\dot{0}\dot{0}00}), \tag{24}$$

$$V_2^1 = -\frac{\beta_\tau \varphi_{0t}}{2\beta^2}(2B^z_{\dot{0}^2 0} + B^z_{\dot{0}^2 0^2}) - \frac{\beta_\tau \varphi_{0t}}{2\beta^2}(C^z_{\dot{0}\dot{0}} - 2C^z_{\dot{0}\dot{0}0} + C^z_{\dot{0}\dot{0}00}) - \frac{\varkappa\varphi_t}{2}(2 - B_{\dot{0}0}) -$$
$$\frac{\varphi_t}{4}(2a_F - 2B_{\dot{0}^2 0} + B_{\dot{0}^2 0^2}) - \frac{\varphi_t}{8}(C_{\dot{0}\dot{0}} - 2C_{\dot{0}\dot{0}0} + C_{\dot{0}\dot{0}00}). \tag{25}$$

Here we denote

$$B_{\dot{0}^2 0} = \int \dot{\omega}_0^2(z)\omega_0(-\eta - z)dz, \quad B_{\dot{0}^2 0^2} = \int \dot{\omega}_0^2(z)\omega_0^2(-\eta - z)dz,$$

$$B^z_{\dot{0}^2 0} = \int z\dot{\omega}_0^2(z)\omega_0(-\eta - z)dz, \quad B^z_{\dot{0}^2 0^2} = \int z\dot{\omega}_0^2(z)\omega_0^2(-\eta - z)dz,$$

$$C_{\dot{0}\dot{0}} = \int \dot{\omega}_0(z)\dot{\omega}_0(-\eta - z)dz, \quad C_{\dot{0}\dot{0}0} = \int \dot{\omega}_0(z)\dot{\omega}_0(z)\omega_0(-\eta - z)dz,$$

$$C_{\dot{0}\dot{0}00} = \int \dot{\omega}_0(z)\dot{\omega}_0(-\eta - z)\omega_0(z)\omega_0(-\eta - z)dz, \quad C^z_{\dot{0}\dot{0}} = \int z\dot{\omega}_0(z)\dot{\omega}_0(-\eta - z)dz,$$

$$C^z_{\dot{0}\dot{0}0} = \int z\dot{\omega}_0(z)\dot{\omega}_0(z)\omega_0(-\eta - z)dz,$$

$$C^z_{\dot{0}\dot{0}00} = \int z\dot{\omega}_0(z)\dot{\omega}_0(-\eta - z)\omega_0(z)\omega_0(-\eta - z)dz,$$

$$a_F = \int F(\omega_0(z))dz, \quad F(\omega_0) = \frac{\omega_0^4}{4} - \frac{\omega_0^2}{2} + \frac{1}{4}. \tag{26}$$

In (16), the coefficients of the derivatives of the δ-functions are given by the formula

$$V_i^2 = \beta^2 \hat{C} - \hat{D}, \qquad i = 1, 2, \tag{27}$$

where we denote

$$\hat{C} = \frac{1}{8}(2a_F - 2B_{\dot{0}^2 0} + B_{\dot{0}^2 0^2} - C_{\dot{0}\dot{0}} + 2C_{\dot{0}\dot{0}0} - C_{\dot{0}\dot{0}00}), \tag{28}$$

$$\hat{D} = \int F\left(\frac{1}{2}(\omega_0(z) + \omega_0(-\eta - z) + \omega_0(z)\omega_0(-\eta - z)) - 1\right)dz. \tag{29}$$

THEOREM. *If Stefan equations (6), equation*

$$\gamma_l^+(-\varphi, t) - \gamma^-(-\varphi, t)B_{11} - \gamma_r^+(\varphi, t) - \gamma^-(\varphi, t)B_{11} + \tag{30}$$
$$\frac{2\beta_\tau \varphi_{0t}}{\beta^2}B^z_{\dot{0}0} - \varphi_t(2 - B_{\dot{0}0}) = 0,$$

Gibbs–Thomson equations (7), and the equation

$$\beta^2\hat{C} - \hat{D} = 0 \tag{31}$$

hold, then the functions $\hat{u}_\varepsilon$,$\hat{\theta}_\varepsilon$ *defined in (13), (14) are the weak asymptotic solution of the phase field system (1), (2).*

Proof. In view of Lemma , if

$$\lim_{\tau\to\pm\infty} W_i^1 = 0, \tag{32}$$

then

$$W_1^1 + W_2^1 = 0. \tag{33}$$

Equation (32) is the Stefan equation (38) as $\tau \to +\infty$. As $\tau \to -\infty$, equations (32) hold automatically. So, we obtain relation (33) that means equation (30).

Analogously, if

$$\lim_{\tau\to\pm\infty} V_i^1 = 0, \tag{34}$$

then

$$V_1^1 + V_2^1 = 0. \tag{35}$$

As $\tau \to +\infty$, equations (34) imply the Gibbs–Thomson conditions (37). As $\tau \to -\infty$, equations (35) hold automatically. So, by Lemma we obtain that equation (33) is true. Taking (24) and (25) into account, we see that equation (33) is true automatically.

Also, we need to assume that $V_i^2 = 0$, $i = 1, 2$, and, therefore, we obtain relation (31).

We note that, in view of (22), we leave out of account the coefficients W_i^2, $i = 1, 2$. □

So, if the assumptions of the Theorem hold, then system (17)–(19), (30) is the regularization of generalization of the Stefan–Gibbs–Thomson problem (4)–(7) for $t \in [0, t_1]$, where $t_1 > t^*$.

3. Dynamics of the free boundaries prior to interaction.

In this section we analyze the weak asymptotic solution (13), (14) of the phase field system (1), (2) for the moments of time those precede the moment of interaction of the free boundaries $\Gamma_{i,t}$, $i = 1, 2$. Taking our assumptions into account, we obtain that prior to interaction $\varphi_1 \to 0$, and, consequently, $\tau \to \infty$, $\rho/\tau \to 1$ as $\varepsilon \to 0$, $t < t^*$.

Now we consider the formula (27) in the limit as $\tau \to \infty$. Clearly, in (28), the terms $C_{\dot{0}\dot{0}}$, $C_{\dot{0}\dot{0}0}$, and $C_{\dot{0}\dot{0}00}$ become zero as $\tau \to \infty$. It is well known that $\ddot{\omega}_0 = -\omega_0 + \omega_0^3$. Consequently, multiplying by $\dot{\omega}_0$ and integrating over $\mathbb{R}_z^1$ the last equation, we obtain the following formula for a_F:

$$2a_F = A_{\dot{0}^2} = \int \dot{\omega}_0^2(z)dz.$$

Obviously, $B_{\dot{0}^20} \to 2a_F$ and $B_{\dot{0}^20^2} \to 2a_F$ as $\rho \to \infty$. Thus, we see that $\hat{C} \to a_F$ as $\rho \to \infty$. A straightforward analysis of the coefficient $\hat{D}$ defined in (29) shows that $\hat{D} \to a_F$ as $\rho \to \infty$.

So, taking into account our assumptions on the function β, the above analysis and the assumption that coefficients (27) are equal to zero, we obtain the equation

$$\beta_0^2 - 1 = 0$$

as $\tau \to \infty$ (i.e., prior to the confluence of the free boundaries). Since, we assume that the function β is positive, the last equation has the single root

$$\beta_0 = 1. \tag{36}$$

Directly from our constructions we obtain that the temperature is continuous and is equal to φ_t at the points of the interfaces $x = \pm\varphi$. Hence, in the limiting Stefan–Gibbs–Thomson problem condition (5) holds.

Now we consider the equation (35) in the prove of the Theorem . As is noted above, this equation is ever true, and we do not include equation (33) in the assumptions of the Theorem . However, this equation is formally contained in system (17)–(19), (30). Nevertheless, for the validity of problem (17)–(19), (30) prior to interaction (while the free boundaries lie on the sufficient large distance one from another), we equate to zero the coefficients (24), (25) as $\tau \to \infty$. In other words, we verify that equation (34) is true. So, we obtain the equations

$$\varkappa\,\gamma(x,t)\big|_{x=\mp\varphi_0} = -A_{\dot{0}^2}\varphi_{0t}, \tag{37}$$

where $\gamma(x,t)$ is the temperature (see (14)). Thus, the Gibbs–Thomson equation (7) holds, where $\alpha = A_{\dot{0}^2}/\varkappa$.

Now we consider equation (30) prior to interaction, or, as is the same, we verify that limits (32) are true as $\tau \to \infty$. Thus, we obtain the two equalities those correspond to the free boundaries $x = \mp\varphi_0(t)$:

$$\left(\gamma^- - \gamma_l^+\right)\big|_{x=-\varphi_0} = -2\varphi_{0t}, \quad \left(\gamma_r^+ + \gamma^-\right)\big|_{x=\varphi_0} = -2\varphi_{0t}. \tag{38}$$

It is clearly seen that equations (38) are identical with the Stefan equations (6) correspondingly for the interfaces $\Gamma_{i,t}$, $i = 1, 2$.

It is easily seen that, as $\tau \to \infty$, the heat equations (17)–(19) pass to the heat equation (4) correspondingly for the domains Ω_t^+, $\Omega_{1,t}^-$ and $\Omega_{2,t}^-$.

So, problem (17)–(19), (30) passes to the Stefan–Gibbs–Thomson problem (4)–(7) prior to confluence.

4. Confluence of the free boundaries.

In this section, we analyze the solution of the phase field system in the case of the Stefan–Gibbs–Thomson problem after confluence of the free boundaries. So, we have $\varphi_{20}(t) - \varphi_{10}(t) < 0$ and, consequently, $\tau \to -\infty$ as $\varepsilon \to 0$.

Taking the fact that the coefficients in (27) are zero and equation (36) into account, we have the formula for $\beta_1(\tau)$:

$$(1+\beta_1)^2 = \frac{\hat{D}}{\hat{C}}. \tag{39}$$

In view of the relation (26) for F, from formula (27), we have $F(z) \geqslant 0$ for $z \in \mathbb{R}^1$, and, consequently, $\hat{D} \geqslant 0$. Analogously, in (27) the function $\hat{C}$ is the leading term of the asymptotics of the expression $(\partial u/\partial x)^2$, and, consequently, $\hat{C}$ is also positive. Using the explicit formula

$$\omega_0(z) = \frac{e^z - e^{-z}}{e^z + e^z},$$

the explicit form of the function $F(z)$ (26), and expressions for $\hat{C}$, $\hat{D}$ (28), (29), we can verify that

$$\lim_{\eta\to-\infty} \hat{C}/e^{2\eta} = \text{const} \neq 0,$$

$$\lim_{\eta\to-\infty} \hat{D}/e^{2\eta} = \text{const} \neq 0.$$

So, the right-hand side of relation (39) is real and positive. This fact means that the single positive root of equation (39) exists. This root is the real bounded function $\beta_1 = \beta_1(\rho)$. Thus, $\beta_1 \to 0$ as $\rho \to \infty$ and $\beta_1 \to \text{const} \neq 0$ as $\rho \to -\infty$.

As is noted in Section 3, equation (5) is true obviously.

The difficulty of the problem with interaction considered here is to find the way of the smooth continuation of the temperature into the domain $t > t^*$ preserving the sign of the derivatives, or, more precisely, we need to smoothly define the temperature in a neighborhood of the point (t^*, x^*). Indicating the way of this continuation, we prove that system (17)–(19), (30) is true for $t > t^*$. In other words, we verify that the asymptotics (13), (14) of the solution of the phase field system (1), (2) is global in time.

We denote $\gamma_1^-(t) = \gamma^-(\varphi, t) = \gamma^-(-\varphi, t)$, and smoothly continue the function $\hat{\gamma}^- = \gamma_1^-(t)$ preserving the sign of the derivative for $[t^*, t_1]$. We assume

$$\hat{\Pi} = \hat{\gamma}^-(x+\varphi)(\varphi - x)/\varphi. \tag{40}$$

We define the function T_ε as the solution of the problem

$$\begin{aligned} &LT_\varepsilon = F(x,t) - \frac{\partial \hat{u}_\varepsilon}{\partial t}, \\ &T_\varepsilon\big|_{t=0} = \hat{\theta}_\varepsilon\Big|_{t=0} - \left\{(\varphi_t + \hat{\Pi})\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi - x}{\varepsilon}\right)\right\}\Big|_{t=0}, \\ &T_\varepsilon\big|_{x=\pm l} = \hat{\theta}_\varepsilon\Big|_{x=\pm l}, \end{aligned} \tag{41}$$

where

$$F(x,t) = -L\left[\hat{\Pi}\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi - x}{\varepsilon}\right)\right].$$

Using the weak asymptotic formulas, we obtain

$$\begin{aligned} F(x,t) - \frac{\partial \hat{u}_\varepsilon}{\partial t} &= \frac{1}{2}\left[\frac{\varphi(\hat{\gamma}_t^-(\varphi^2 - x^2) + \hat{\gamma}^-\varphi_t) - \varphi_t\hat{\gamma}^-(\varphi^2 - x^2)}{\varphi^2} + \frac{\hat{\gamma}^-}{\varphi}\right] B_{11}\times \\ &(H(\varphi - x) - H(-x-\varphi)) + \left[\hat{\gamma}^-\delta(x-\varphi) + \hat{\gamma}^-\delta(x+\varphi)\right] B_{11} + \\ &\frac{\varphi_t}{2}(2 - B_{\dot{0}0})[\delta(x-\varphi) + \delta(x+\varphi)] - \frac{2\beta_\tau\varphi_{0t}}{\beta^2} B_{\dot{0}0}^z + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \end{aligned} \tag{42}$$

and

$$\begin{aligned} -\left\{(\varphi_t + \hat{\Pi})\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi - x}{\varepsilon}\right)\right\}\Big|_{t=0} &= \\ \{\varphi_t B_{11}(H(\varphi - x) - H(-x-\varphi)\}\big|_{t=0} + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{43}$$

Now we can see that if we know the function φ globally it time, then the functions $F(x,t)$ and $\partial\hat{u}_\varepsilon/\partial t$ in the right-hand side of the heat equation in (41) are globally defined as well. Hence, problem (41) allows us to define the function T_ε for $t \geqslant t^*$. We can verify that different continuations of $\hat{\gamma}$ for $t \geqslant t^*$ lead to changing of the order $\mathcal{O}(\varepsilon)$ for the function $F(x,t)$. This fact follows from Lemma 3.

Obviously, the equality

$$\hat{\theta}_\varepsilon = T_\varepsilon + \hat{\Pi}\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi-x}{\varepsilon}\right) \tag{44}$$

holds (see (41)).

Since in the problem considered the domain $\Omega_t^- = [-\varphi, \varphi]$ exists only for $t < t^*$ (as $\varphi < 0$ for $t > t^*$ then $\hat{\Pi}\omega_1\left(\frac{x+\varphi}{\varepsilon}\right)\omega_1\left(\frac{\varphi-x}{\varepsilon}\right) = \mathcal{O}(\varepsilon^N)$ for $t > t^*$) then $\hat{\theta}_\varepsilon = T_\varepsilon + \mathcal{O}(\varepsilon)$ for $t > t^*$. Clearly, in view of (41) we have $T_\varepsilon \in \mathbb{C}^\infty$ for $t > t^*$. Taking (14) into account, we obtain the expression for T_ε:

$$\begin{aligned} T_\varepsilon = & B_{\dot{1}1}\hat{T}^-[H(\varphi - x) - H(-x-\varphi)] + \\ & T_l^+ H(-x-\varphi) + T_r^+ H(\varphi - x) + \varphi_t + \mathcal{O}(\varepsilon), \end{aligned} \tag{45}$$

where $\hat{T}^- = \hat{\Pi} - T^-$, $\hat{T}^- \in \mathbb{C}^1([-l,l])$ uniformly in t. Denoting the leading term of the weak asimptotics of T_ε by T_0, we have

$$T_0\big|_{x=\pm\varphi} = \left\{B_{\dot{1}1}\hat{T}^-[H(\varphi-x) - H(-x-\varphi)]\right\}\Big|_{x=\pm\varphi} + \varphi_t = \varphi_t. \tag{46}$$

So, problem (41) allows us to define the "global" solution that, first, describe the temperature in the domain Ω for $t < t^*$ (see formula (44)) and, second, is independent of topology of the domain. The last fact means that the "global" solution of problem (41) is defined for $t \geqslant t^*$, i.e., after confluence of free boundaries.

The facts mentioned above are true under the assumption that the function φ is specified for $t > t^*$. It seems that equation (30) allows us to specify the function φ for $t > t^*$. However, this equation contains the unknown functions $\gamma_{l,r}^+(\mp\varphi, t)$ those mean the left (right) derivatives of the temperature at the points $x = \mp\varphi$. We note that, in view of (45), the functions $\gamma_{l,r}^+$ can be find from (41) as the jumps of the derivatives of the function $(T_\varepsilon)_x$ at the points $x = \mp\varphi$. It is easy to verify that the equations

$$\gamma_l^+ = \frac{\varphi_t}{2}(2 - B_{\dot{0}0}) + \hat{\gamma}^- B_{\dot{1}1} - \frac{2\beta_\tau\varphi_{0t}}{\beta^2}B_{\dot{0}0}^z, \tag{47}$$

$$\gamma_r^+ = -\frac{\varphi_t}{2}(2 - B_{\dot{0}0}) - \hat{\gamma}^- B_{\dot{1}1} + \frac{2\beta_\tau\varphi_{0t}}{\beta^2}B_{\dot{0}0}^z \tag{48}$$

hold. It is easy seen that these equations exactly imply that the coefficients of $\delta(x \pm \varphi)$ are zero. But equation (30) is obtained by equality of the sum of these coefficients to zero; therefore equations (47), (48) imply (30). We must note that in view of symmetry our problem the equations (47) and (48) are identical. We also note that, in view of (47), (48), $\gamma_{l,r}^+(\mp\varphi) \to 0$ for $t > t^*$. This fact means smoothness of the function T_ε for $t > t^*$ again.

Equation (47) (or (48)) contains the two unknown functions $\gamma_l^+\big|_{x=-\varphi}$ (or $\gamma_r^+\big|_{x=\varphi}$) and φ, therefore the construction of the single formula for the weak asymptotic solution does not lead to success. We assume, as usual,

$$\varphi = \varphi_0 + \varphi_0\varphi_1(\tau, t), \quad \gamma_l^+\big|_{x=-\varphi} = \gamma_{l0}^+ + \gamma_{l1}^+(\tau, t), \tag{49}$$

where $\varphi_1 \to 0$, $\gamma_{l1}^+ \to 0$ as $\tau \to +\infty$ (prior to interaction) are smooth uniformly bounded functions, and its derivatives decreases in τ faster than $|\tau|^{-1}$.

As $\tau \to -\infty$, taking (47) into account, we obtain that the functions γ_{l0}^+, φ_0, γ^- form the Stefan equation (38). Let now the function φ_1 is an arbitrary function and it satisfies to the conditions formulated above and this function is so that $|1+\varphi_1| \geqslant C > 0$ for all $\tau \in (-\infty, +\infty)$, $t \in [0, t_1]$. Then from (47) we obtain the relation

$$\gamma_{l1}^+\big|_{x=-\varphi} = \varphi_{0t}\left(1-(\tau\varphi_1)_\tau\right)\left(1-\frac{B_{\dot{0}0}}{2}\right) - 2\varphi_{0t} + (B_{\dot{1}1}-1)\hat{\gamma}^- - \frac{2\beta_\tau\varphi_{0t}}{\beta^2}B_{\dot{1}1}^z, \tag{50}$$

and we can define the function $\gamma_{r1}^+\big|_{x=\varphi}$ analogously. In particular, the assumption about growth in τ of the derivatives is followed from (50).

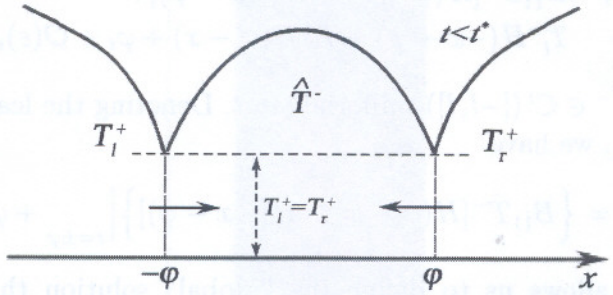

Figure 2. Structure of the temperature.

The numerical simulation shows (see Fig.1 b) that the temperature is the smooth function at the moment of confluence. The theoretical explanation of this fact is the problem (41). Indeed, the function that satisfies to a heat equation must has the smooth coordinate derivatives.

Moreover, from (47) and (48) we can see that after interaction of free boundaries ($\tau \to -\infty$) the functions $\gamma_{l,r}^+ \to 0$. It is easily seen that in (40) the function $\hat{\Pi}$ change the sign at the moment of confluence, and $\hat{\Pi} < 0$, $t > t^*$. From the numerical results in Fig.1 b) we see that, at the moment of confluence $t = t^*$, the temperature change the profile. This effect appears as the temperature drop, see Fig.1 b). An explanation of this temperature profile changing, as was mentioned above, follows from (40) and (44). For clarity, we present the Fig.2, where the temperature is shown schematically.

5. The techniques of the weak asymptotics method.

In this section, using as example the phase field system (1), (2), we demonstrate the techniques of the weak asymptotic method.

The following lemma is the main technical result that we use in our calculations.

Let $\Gamma_t = \{x - \varphi(t) = 0\}$, $x \in \mathbb{R}$, where $\varphi(t)$ is some smooth function, let $\omega(z) \in \mathbb{S}$ ($\mathbb{S}$ is the Schwartz space), and let $\beta = \beta(t) > 0$.

LEMMA 1. *For any test function $\zeta(x)$, the relation*

$$\frac{1}{\varepsilon}\left\langle \omega\left(\beta\frac{x-\varphi(t)}{\varepsilon}\right), \zeta(x)\right\rangle = \frac{1}{\beta}A_\omega\zeta(\varphi) + \mathcal{O}(\varepsilon) \tag{51}$$

holds. Here $A_\omega = \int_{-\infty}^{\infty} \omega(z)dz$.

Proof. The relation in the right-hand side of Eq. (51) can be rewritten in the form

$$\frac{1}{\varepsilon}\int \omega\left(\beta\frac{x-\varphi(t)}{\varepsilon}\right)\zeta(x)dx = \zeta(\varphi)\int \omega(z)dz + \mathcal{O}(\varepsilon).$$

To obtain this relation, we must make the change of variable $z = \beta(x-\varphi)/\varepsilon$, and apply the Taylor formula to the integrand at the point $x = \varphi$. According to the definition the last integral is the action of the function $A_\omega\delta(x-\varphi)$ to the test function ζ. □

If we want to consider linear combinations of generalized functions with accuracy $\mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha)$, then we need to modify the notion of linear independence. This modification plays the key role in the considerations related to problems with interaction of solitons.

Indeed, let $\phi_1 \neq \phi_2$ be functions independent of x. We consider the relation

$$g_1\delta(x-\phi_1) + g_2\delta(x-\phi_2) = \mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha), \quad \alpha > 0, \tag{52}$$

where the functions g_i are independent of ε. Clearly, we obtain the relations

$$g_i = \mathcal{O}(\varepsilon^\alpha), \quad i = 1, 2.$$

Taking our assumptions into account, we have

$$g_i = 0, \quad i = 1, 2.$$

Everything is different if we assume that the coefficients g_i can depend on ε. Here we consider only a special case of such dependence. Namely, let

$$g_i = A_i + S_i(\Delta\phi/\varepsilon), \quad i = 1, 2, \tag{53}$$

where A_i are independent of ε and, as $|\sigma| \to \infty$, the $S_i(\sigma)$ decrease sufficiently fast.

LEMMA 2. *Let the estimate*

$$|\sigma S_i(\sigma)| \leqslant \text{const}, \quad i = 1, 2$$

hold. Then the relations

$$A_1 = 0, \quad A_2 = 0, \quad S_1 + S_2 = 0 \tag{54}$$

are follow from relation (52) for $\alpha = 1$.

Proof. Using in (52) the Teylor formula and taking (53) into account, we have

$$[S_1\zeta(\phi_1)) + S_2\zeta(\phi_2))] = S_1\zeta(\phi_1)) + S_2\zeta(\phi_1)) + S_2(\phi_2-\phi_1)\zeta'(\phi_1+\mu\phi_2), \ 0 < \mu < 1.$$

We see that

$$S_2(\Delta\phi/\varepsilon)(\phi_2-\phi_1) = \{-\sigma S_2(\sigma)\}|_{\sigma=\Delta\phi/\varepsilon}\cdot\varepsilon = \mathcal{O}(\varepsilon),$$

since the function $\sigma S_2(\sigma)$ is uniformly bounded in $\sigma \in \mathbb{R}_1$.

So, we can rewrite relation (52) in the form

$$A_1\zeta(\phi_1) + A_2\zeta(\phi_2) + (S_1+S_2)\zeta(\phi_1) = \mathcal{O}(\varepsilon).$$

Thus, since the coefficients A_i are independent of ε, we obtain the statement of the lemma. □

COROLLARY 1. *Let, in relation (52),*

$$g_i(z) \in \mathcal{C}^\infty,\ g_i = g_i\left(\frac{\Delta\phi}{\varepsilon}\right),\ \left.\frac{\partial g_i(z)}{\partial z}\right|_{|z|\to\infty} = \mathcal{O}\left(|z|^{-N}\right), \qquad N > 0,\ i = 1, 2.$$

Then we have the equality

$$g_1 + g_2 = 0.$$

Proof. We write the functions g_i in the form

$$g_i(z) = g_i^- + \omega(z)(g_i^+ - g_i^-) + g_i(z) - g_i^- - \omega(z)(g_i^+ - g_i^-), \tag{55}$$

where $\omega \in \mathbb{C}^\infty$, $\omega(-\infty) = 0$, $\omega(\infty) = 1$ and $\omega^{(\alpha)} \in \mathbb{S}$ for $\alpha > 0$, $g_i^\pm = \lim_{z\to\pm} g_i$. We note that the expressions in the parenthesis in (55) behave in the same way as the functions S_i in (54). Relations (52) are equivalent to the following limits:

$$\lim_{z\to\pm\infty} g_i(z) = 0, \qquad i = 1, 2. \tag{56}$$

Therefore, if (56) hold, then we obtain the situation of the Lemma 2 for $A_i = 0$. □

LEMMA 3. *Let $f(t) \in \mathbb{C}^1$, $f(t_0) = 0$, and let $f'(t_0) \neq 0$. Let $g(t,\tau)$ the locally uniformly satisfy the estimates*

$$|\tau g(t,\tau)| \leqslant \text{const}, \quad |\tau g'(t,\tau)| \leqslant \text{const}, \quad -\infty < \tau < \infty,$$

and let $g(t_0,\tau) = 0$. Then the inequality

$$|g\left(t, f(t)/\varepsilon\right)| \leqslant \varepsilon C_{\hat t}$$

holds on any interval $0 \leqslant t \leqslant \hat t$ that does not contain zeros of the function $f(t)$ except t_0, where $C_{\hat t} = \text{const}$.

Proof. The fraction $f(t)/(t-t_0)$ is locally bounded in t. The fraction $\tau g(t,\tau)/(t-t_0)$ is also locally bounded. We have

$$g\left(t, f(t)/\varepsilon\right) = \varepsilon \cdot \frac{g\left(t, f(t)/\varepsilon\right)}{(t-t_0)} \cdot \frac{f(t)}{\varepsilon} \cdot \frac{t-t_0}{f(t)}.$$

By the assumptions of the lemma, on the interval under study, the last multiplier on the right-hand side is bounded, while the product of the intermediate multiplier and the first multiplier (without ε) is bounded in view of the properties of the function $g(t,\tau)$. □

COROLLARY 2. *Suppose that the estimates in the assumptions of Lemma 3 hold for $0 \leqslant \tau < \infty$ ($-\infty < \tau \leqslant 0$). Then the statement of Lemma 3 holds on any interval $[t_0, \hat t]$ that does not contain zeros of the function $f(t)$, and $\text{sign}\hat t = \text{sign} f(t)$, $t \in [t_0, \hat t]$.*

According to Definition 1, we must substitute asimptotics (13), (14) into system (15), (16). As a result, equation (15) contains the terms of the several types. We consider the terms contained the function $\omega_0(\beta(-x-\varphi)/\varepsilon)$:

$$\int f(x,t)\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\zeta(x)dx =$$
$$\int \omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\left(\frac{d}{dx}\int_{-\infty}^{x} f(y,t)\zeta(y)dy\right)dx =$$
$$\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\int_{-\infty}^{x} f(y,t)\zeta(y)dy\Big|_{-\infty}^{\infty} +$$
$$\frac{\beta}{\varepsilon}\int \dot{\omega}_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\left(\int_{-\infty}^{x} f(y,t)\zeta(y)dy\right)dx =$$
$$\int \zeta(x)dx - \int \dot{\omega}_0(z)dz\int_{-\infty}^{-\varphi} f(x,t)\zeta(x)dx + \mathcal{O}(\varepsilon),$$

where $f(x,t)$ is some smooth function, and $\zeta(x)$ is the test function. (We recall that the integrals are taken over $\mathbb{R}^1$). The last integral is obtained as the result of, first, using of change of variable $z=\beta(-x-\varphi)/\varepsilon$, and, second, using of the Teylor formula at the point $x=-\varphi$. We note that, according to Lemma 1, our transformations are true. As a result, we have

$$f(x,t)\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right) = f(-\varphi,t)(1+2H(-x-\varphi)) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{57}$$

where $H(z)$ is the Heaviside function. Using the same method, in (13), we obtain the expressions for the terms contained the function $\omega_0(\beta(x-\varphi)/\varepsilon)$

$$f(x,t)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right) = f(\varphi,t)(1-2H(\varphi-x)) + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{58}$$

As we noted above, the principle moment of the weak asymptotics method is the transformation of nonlinearity (product of the two functions $\omega_0(\beta(-x-\varphi)/\varepsilon)$ and $\omega_0(\beta(x-\varphi)/\varepsilon)$, see formula (13)) to the linear combination of the Heaviside functions (see Lemma 2). Namely, we have

$$\int f(x,t)\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right)\zeta(x)dx =$$
$$\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right)\int_{-\infty}^{x} f(y,t)\zeta(y)dy\Big|_{-\infty}^{\infty} +$$
$$\frac{\beta}{\varepsilon}\int \dot{\omega}_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right)\left(\int_{-\infty}^{x} f(y,t)\zeta(y)dy\right)dx-$$
$$\frac{\beta}{\varepsilon}\int \dot{\omega}_0\left(\beta\frac{x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\left(\int_{-\infty}^{x} f(y,t)\zeta(y)dy\right)dx =$$
$$\int f(x,t)\zeta(x)dx + \int \dot{\omega}_0(z)\omega_0(-z-\eta)dz\int_{-\infty}^{-\varphi} f(x,t)\zeta(x)dx-$$
$$\int \dot{\omega}_0(z)\omega_0(-z-\eta)dz\int_{-\infty}^{\varphi} f(x,t)\zeta(x)dx + \mathcal{O}(\varepsilon).$$

Analogously, as we calculate for formulas (57) and (58), in the two last integrals we correspondingly change the variables $z = \beta(\mp x - \varphi)/\varepsilon$, and we use the Taylor formula at the points $x = \mp\varphi$. Thus, we have

$$f(x,t)\omega_0\left(\beta\frac{-x-\varphi}{\varepsilon}\right)\omega_0\left(\beta\frac{x-\varphi}{\varepsilon}\right) = \tag{59}$$
$$1 + B_{\dot{0}0}\left[f(-\varphi,t)H(-x-\varphi) - f(\varphi,t)H(\varphi-x)\right] + \mathcal{O}_{\mathcal{D}'}(\varepsilon),$$

where $B_{\dot{0}0}(\eta) = \int \dot{\omega}_0(z)\omega_0(-z-\eta)dz$, $\eta = 2\beta\rho$, $\rho = \varphi/\varepsilon$.

Using the technique demonstrated above, it is easy to verify that the calculation of $L\hat{\theta}_0 + \partial\hat{u}_0/\partial t = \mathcal{O}_{\mathcal{D}'}(\varepsilon)$ (see (1)) leads to the linear combination of general functions (15). Analogously, we obtain the linear combination (16).

We note that the indicated calculations (in particular, the calculations of the derivatives $\partial\hat{u}_0/\partial t$ and $\partial\hat{\theta}_0/\partial t$) lead to existence of the terms those contain products of the Heaviside functions and the integrals $\left(B_{\dot{j}j}\right)_t$, $j = 0, 1$. The expressions of this type we transform as in following

$$\left(B_{\dot{j}j}\right)_t f[H(-x-\varphi) - H(\varphi-x)] =$$
$$\frac{2}{\varepsilon}B'_{\dot{j}j}(\eta)(\beta\rho)_\tau f[H(-x-\varphi) - H(\varphi-x)] =$$
$$2\rho B'_{\dot{j}j}(\eta)(\beta\rho)_\tau f[\lambda\delta(x+\varphi) + (1-\lambda)\delta(x-\varphi)] + \mathcal{O}_{\mathcal{D}'}(\varepsilon),$$

where f is some function, and $0 \leqslant \lambda \leqslant 1$. Taking the symmetry of the problem considered into account, we assume $\lambda = 1/2$.

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* *MTUCI,*
Aviamotornaya, 8A,
111024 Moscow, Russia,
† *MIEM,*
B.Trekhsvyatitel'skiy d.1-3/12, b.8
109028 Moscow, Russia
pm@miem.edu.ru